Probability: The Foundation of Uncertainty

PSC4375: Week 9

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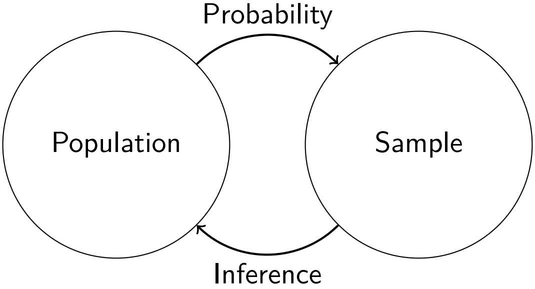
Learning about populations



Probability: formalize the uncertainty about how our data came to be
Inference: learning about the population from a sample of data

Why probability?

- Probability quantifies chance variation or uncertainty in outcomes.
    - It might rain or be sunny today, we don't know which.
- We estimated a treatment effect of 7.2, but what if we reran history?
    - Weather changes $\rightsquigarrow$ slightly different estimated effect.
- Statistical inference is a **thought experiment** about uncertainty.
    - Imagine a world where the treatment effect were 0 in the population.
    - What types of estimated effects would we see in this world by chance?
- Probability to the rescue!

Sample spaces & events

- To formalize chance, we need to define the set of possible outcomes.
- **Sample space**: Ω the set of possible outcomes.
- **Event**: any subset of outcomes in the sample space

Example: gambling

- A standard deck of playing cards has 52 cards:
    - 13 rank cards: (2,3,4,5,6,7,8,9,10,J,Q,K,A)
    - in each of 4 suits: (♣,♠,♡,♢)
    - Hypothetical trial: pick a card, any card.
        - Uncertainty: we don't know which card we're going to get.
    - One possible outcome: picking a 4♣
    - Sample space:

2♣	3♣	4♣	5♣	6♣	7♣	8♣	9♣	10♣	*J*♣	*Q*♣	*K*♣	*A*♣
2♠	3♠	4♠	5♠	6♠	7♠	8♠	9♠	10♠	*J*♠	*Q*♠	*K*♠	*A*♠
2♡	3♡	4♡	5♡	6♡	7♡	8♡	9♡	10♡	*J*♡	*Q*♡	*K*♡	*A*♡
2♢	3♢	4♢	5♢	6♢	7♢	8♢	9♢	10♢	*J*♢	*Q*♢	*K*♢	*A*♢

- An event: picking a Queen, {Q♣,Q♠,Q♡,Q♢}

What is probability?

- The probability $\mathbb{P}(A)$ represents how likely event A occurs.
- If **all outcomes equally likely**, then:

$$\mathbb{P}(A) = \frac{\text{number of elements in } A}{\text{number of elements in } \Omega}$$

- Example: randomly draw 1 card:
  - probability of drawing 4♣: $\frac{1}{52}$
  - probability of drawing any ♣: $\frac{13}{52}$
- Same math, but different interpretations:
  - **Frequentist**: $\mathbb{P}()$ reflects relative frequency in a large number of trials.
  - **Bayesian**: $\mathbb{P}()$ are subjective beliefs about outcomes.
- Not our fight $\rightsquigarrow$ stick to frequentism in this class

Probability axioms

- Probability quantifies how likely or unlikely events are.
- We'll define the probability $\mathbb{P}(A)$ with three axioms:

1. (Nonnegativity) $\mathbb{P}(A) \geq 0$ for every event A
2. (Normalization) $\mathbb{P}(\Omega) = 1$
3. (Addition Rule) If two events A and B are mutually exclusive

$$\mathbb{P}(A \text{ or } B) = \mathbb{P}(A) + \mathbb{P}(B).$$

Gambling 102

- What is $\mathbb{P}(\text{Q card})$ if a single card is randomly selected from a deck?
  - "randomly selected" $\rightsquigarrow$ all cards have prob. $1/52$
- "4 card" event $= \{\text{Q}\clubsuit \text{ or } \text{Q}\spadesuit \text{ or } \text{Q}\heartsuit \text{ or } \text{Q}\diamondsuit\}$
- Union of mutually exclusive events $\rightsquigarrow$ use addition rule
  - $\rightsquigarrow \mathbb{P}(\text{Q card}) = \mathbb{P}(\text{Q}\clubsuit) + \mathbb{P}(\text{Q}\spadesuit) + \mathbb{P}(\text{Q}\heartsuit) + \mathbb{P}(\text{Q}\diamondsuit) = \frac{4}{52}$

Useful probability facts

- Probability of the complement: $\mathbb{P}(\text{not } A) = 1 - \mathbb{P}(A)$
  - Probability of **not** drawing a Queen is $1 - \frac{4}{52} = \frac{48}{52}$
- **General addition rule** for any events, A and B:

$$\mathbb{P}(A \text{ or } B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \text{ and } B)$$

- Probability of drawing Queen or ♠
- $\frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52}$

Conjunction fallacy

Linda is 31 years old, single, outspoken, and very bright. She majored in philosophy. As a student, she was deeply concerned with issues of discrimination and social justice, and also participated in anti-nuclear demonstrations.

- What is more probable?
  1. Linda is a bank teller?
  2. Linda is a bank teller and is active in the feminist movement?
- Famous example of the **conjunction fallacy** called the Linda problem
  - Majority of respondents chose 2, but this is impossible!
- **Law of total probability** for any events A and B:

$$\mathbb{P}(A) = \mathbb{P}(A \text{ and } B) + \mathbb{P}(A \text{ and not } B)$$

$$\mathbb{P}(\text{teller and feminist}) = \mathbb{P}(\text{teller}) - \mathbb{P}(\text{teller and not feminist})$$

Law of total probability

	Democrats	*Republicans*	*Independents*	*Total*
Men	29	43	2	74
Women	16	10	0	26
Total	45	53	2	100

- What's the probability of randomly selecting a woman senator?

$$\mathbb{P}(\text{woman}) = \mathbb{P}(\text{woman and a Democrat}) + \mathbb{P}(\text{woman and not a Democrat})$$

$$= \frac{16}{100} + \frac{10}{100} = \frac{26}{100}$$

Break time!

Conditional probability

- If we know that B has occurred, what is the probability of A?
  - Conditioning our analysis on B having occurred.
- Examples:
  - Probability of two states going to war if they are both democracies?
  - Probability of a judge issuing a pro-choice ruling if they have daughters?
  - Probability of a coup in a country if it has a presidential system?
- Conditional probability extremely useful for data analysis.

Conditional probability definition

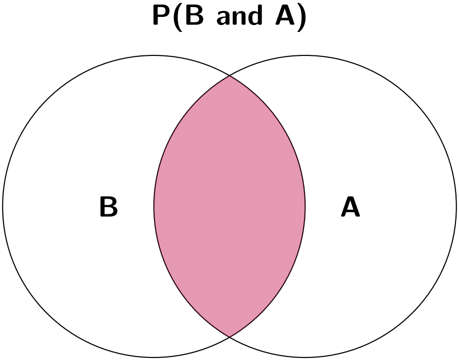
- Definition: if $\mathbb{P}(B) > 0$ then the **conditional probability** of A given B is

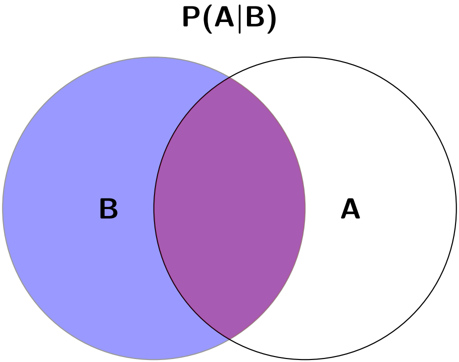
$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \text{ and } B)}{\mathbb{P}(B)}$$

- How often A and B occur divided by how often B occurs.
- **WARNING!** $\mathbb{P}(A|B)$ does **not**, in general, equal $\mathbb{P}(B|A)$
  - $\mathbb{P}(\text{smart}|\text{in QSS})$ is high
  - $\mathbb{P}(\text{in QSS}|\text{smart})$ is low
  - There are many many smart people who are not in this class (tell your friends)
- If all outcomes equally likely:

$$\mathbb{P}(A|B) = \frac{\text{number of outcomes in both } A \text{ and } B)}{\text{number of outcomes in just } B}$$

Conditional probability




P(A|B)
B
A

US Senate example

	Democrats	*Republicans*	*Independents*	*Total*
Men	29	43	2	74
Women	16	10	0	26
Total	45	53	2	100

- Choose one senator at random from this population
- What is the probability of choosing a woman?
  - $\mathbb{P}(\text{Woman}) = \frac{26}{100} = 0.26$
- What is the probability of choosing a Republican who is a woman?
  - $\mathbb{P}(\text{Woman} \mid \text{Rep.}) = \frac{10/100}{53/100} \approx 0.19$

Conditional probability rules

- Multiplication rule:

$$\mathbb{P}(A \text{ and } B) = \mathbb{P}(A|B)\mathbb{P}(B) = \mathbb{P}(B|A)\mathbb{P}(A)$$

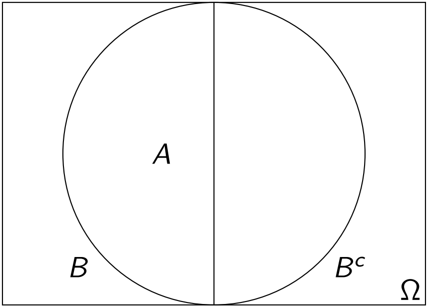
Multiplication rule example

	Democrats	*Republicans*	*Independents*	*Total*
Men	29	43	2	74
Women	16	10	0	26
Total	45	53	2	100

- Draw the names of two senators from a hat
- What's the probability that we draw two women?
  - Let W_1 and W_2 be the events that 1st and 2nd draws are women
  - We could make a list of all possible pairs to draw and count them...
  - Or we could just use the multiplication rule:

$$\mathbb{P}(W_1 \text{ and } W_2) = \mathbb{P}(W_1)\mathbb{P}(W_2|W_1)$$

Law of total probability



- Conditional probability lets us restate the law of total probability
- **Law of total probability**:

$$\begin{aligned}\mathbb{P}(A) &= \mathbb{P}(A \text{ and } B) + \mathbb{P}(A \text{ and not } B) \\ &= \mathbb{P}(A|B)\mathbb{P}(B) + \mathbb{P}(A| \text{ not } B)\mathbb{P}(\text{not } B)\end{aligned}$$

Sampling and independence

- Sampling > 1 with replacement: **independent draws**
    - Randomly draw 1 senator, note the name, then put it back in the hat.
    - Shuffle, randomly draw 2nd senator, note the senator.
    - First draw doesn't affect second $\rightsquigarrow$ independence
    - Sampling > 1 without replacement: **dependent draws**
        - Randomly draw 1 senator, note the name, leave it out.
        - Randomly draw 2nd senator from the remaining 99 senators.
        - First draw affects the probability of the second.